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Combining monotonicity and strong equity: construction and representation of orders on infinite utility streams

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Abstract This paper studies the nature of social welfare orders (SWO) on infinite utility streams, satisfying the efficiency principle known as monotonicity and the consequentialist equity principle known as strong equity. It provides a complete characterization of domain sets for which there exists such a SWO which is in addition representable by a real valued function. It then shows that for those domain sets for which there is no such SWO which is representable, the existence of such a SWO necessarily entails the existence of a non-Ramsey set, a non-constructive object.

Keywords Social welfare orders · Monotonicity · Strong Equity · Representation · Construction · Non-Ramsey set

JEL Classification D60 · D70 · D90

1 Introduction

The conflict between principles of equity and efficiency in the evaluation of infinite utility streams[1] has been discussed extensively in the literature.

The analysis of such conflicts depends, of course, on the precise nature of the efficiency and equity principles that are imposed. In this paper, we focus on the weakest

[1] We use the standard framework in which the space of infinite utility streams is given by $X = Y^{\mathbb{N}}$, where Y is a non-empty set of real numbers, and $\mathbb{N}$ the set of natural numbers.

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efficiency principle, which is generally referred to as monotonicity (M). This efficiency concept is incontrovertible as it only requires that if no one is worse off (in utility stream x compared to y), then the society as a whole should not be worse off (in utility stream x compared to y).

The equity concept that we examine in this paper (called the Strong Equity (SE) axiom) belongs to the class of *consequentialist* equity concepts, dealing with situations in which the *distribution* of utilities of generations has changed in specific ways. The SE axiom is a strong form of the equity axiom of Hammond (1976)[2] and involves comparisons between two utility streams (x and y) in which all generations except two have the same utility levels in both utility streams. Regarding the two remaining generations (say, i and j), one of the generations (say i) is better off in utility stream x, and the other generation (j) is better off in utility stream y, thereby setting up a conflict. The axiom states that if for both utility streams, it is generation i which is worse off than generation j (this, of course, requires us to make intergenerational comparisons of utilities), then generation i should be allowed (on behalf of the society) to choose between x and y. That is, x is socially preferred to y, since generation i is better off in x than in y.

The equity axiom of Hammond is one of the key consequentialist equity concepts, the other being the Pigou–Dalton transfer principle.[3] Note that under the situation described in the previous paragraph, Hammond Equity would make the weaker statement that x is at least as good as y. Since our efficiency concept (monotonicity) does not require sensitivity, the combination of Hammond Equity and monotonicity would clearly be satisfied by the trivial social welfare order (SWO) which considers all utility streams to be socially indifferent. Thus, having accepted monotonicity as the incontrovertible efficiency concept, it is natural to focus on SE, the stronger form of the equity axiom of Hammond, and to ask whether these two axioms are compatible.

Bossert et al. (2007) have shown (when $Y = \mathbb{R}$) that there exist SWO on infinite utility streams which satisfy Hammond Equity (referred to as " Equity Preference"), finite anonymity and strong Pareto. Under strong Pareto, Hammond Equity and SE are equivalent. So, in particular, we know that there exist social welfare orders on infinite utility streams which satisfy the SE axiom and monotonicity. Our paper is devoted to understanding the nature of such SWO. Clearly such orders can be useful in decision making provided they can be *represented* by a real-valued function, or at least (even if they do not have a real-valued representation) if they can be *constructed*.

The existence of SWO, satisfying Hammond Equity and strong Pareto, is established in Bossert et al. (2007) by using the variant of Szpilrajn's Lemma given in Arrow

[2] The SE axiom was introduced by d'Aspremont and Gevers (1977), who referred to it as an *Extremist Equity Axiom*.

[3] Hammond Equity has several variations which have been discussed in the literature. SE and *Hammond Equity for the Future* (see Asheim et al. 2007; Banerjee 2006) are notable variations. *Altruistic Equity* is a variation of the Pigou–Dalton transfer principle. Hara et al. (2008) have shown the incompatibility of upper semicontinuous acyclic binary relations with the Pigou–Dalton transfer principle. On the other hand, Sakamoto (2012) has established the possibility of a social welfare function, satisfying the Pigou–Dalton transfer principle, by using the axiom of choice. More recently, Dubey and Mitra (2013b) have shown that the existence of any social welfare order (SWO), satisfying the Pigou–Dalton transfer principle, necessarily involves a non-constructive device.

(1963), a non-constructive device.[4] This, of course, leaves open the question of whether such SWO can be constructed. More pertinent from the standpoint of the current investigation, it leaves open the question of whether SWO, satisfying the SE axiom and monotonicity, can be constructed.

Turning to the representation issue, it is known (see Alcantud and Garcia-Sanz (2013)) that any SWO satisfying Hammond Equity and strong Pareto cannot be represented by a real-valued function, if the domain set (Y) consists of at least four distinct elements. That is, an impossibility result arises as soon as we admit a situation in which Hammond Equity can play a role in ranking two utility streams.[5] In particular, then, the same impossibility result arises for any SWO satisfying SE and strong Pareto. But, this leaves open the question of whether SWO, satisfying the SE axiom and monotonicity, can be represented. The following theorem is our contribution on these two open questions.

Theorem *(a) There exists a social welfare function that combines the strong equity axiom and monotonicity if and only if the cardinality of Y is at most five. In this case, it is possible to explicitly construct such a social welfare function. (b) If the cardinality of Y is at least six, then it is impossible to construct a SWO that combines the strong equity axiom and monotonicity.*

Tackling the representation issue first, we show that there exist social welfare function (SWF) satisfying the SE axiom and monotonicity if the domain set (Y) has at most five distinct elements. For Y containing more than five elements, we prove an impossibility result using the technique introduced by (Basu and Mitra 2003, Theorem 1).[6]

Our possibility result is established by an *explicit construction* of the SWF (when Y consists of at most five distinct elements), which is of considerable interest. In case $Y = \{a, b, c, d, e\}$, listed in increasing order, the SWF that establishes the positive result is defined as follows. For a given utility stream x the population $\mathbb{N}$ is partitioned into the extreme poor $(i|x_i = a)$, the poor $(i|x_i = b)$, and non-poor $(i|x_i \geq c)$. If everyone is non-poor in utility stream x, then the social welfare in x is strictly positive and is a discounted sum of the period utilities $(x_i - a)$. If x contains at least one generation who is poor or extreme poor, then the social welfare in x is strictly negative and takes into account every poor and extreme poor generation but completely ignores all information about the non-poor generations, if any. [This, of course, means that the social welfare function violates strong Pareto, but it will satisfy monotonicity, which is our efficiency requirement]. Also the social welfare function assigns more negative weight to the extreme poor compared to the poor.

In order to apply the SE axiom to utility streams x and y, four distinct utility levels are involved. At least one generation is poor or extreme poor and at most three

[4] Lauwers (2010) defines the ultrafilter-leximin order that satisfies strong Pareto, finite anonymity, and SE (referred to by him as "Hammond Equity"). Observe that the free ultrafilter used in the definition is also a non-constructive device.

[5] In case (Y) is not rich enough (i.e., consists of less than four distinct elements), any strong Paretian social welfare function (SWF) satisfies Hammond Equity or SE trivially.

[6] Since establishing this complete characterization, we have become aware that the impossibility part of the characterization is similar to a result established by Alcantud (2013a), using an equity concept slightly stronger than SE, when Y has four distinct elements satisfying some additional constraints.

generations are non-poor. In the case where three non-poor generations are involved, there is a jump from negative to positive welfare evaluation if remaining generations are non-poor. If not all of the remaining generations are non-poor, the differential weights assigned to extreme poor and poor generations in the SWF ensures that the SE property holds. This argument also applies in case only two generations are non-poor in the SE comparison.

The low cardinality requirement (on the set Y) for our possibility result should not obscure the fact that the SWF is defined on X, which still consists of an *uncountably infinite* number of distinct utility streams. Further, from the practical policy point of view, we do quite often deal with a low cardinality of the domain set Y, using distinct utility levels to distinguish (for instance) the "rich", the "poor", the "upper middle class", and the "lower middle class".

Turning to the construction issue, the question we address is the following. When the domain set (Y) has more than five distinct elements (a situation in which we know that there is no social welfare function satisfying the SE axiom and monotonicity) is it possible to construct a SWO satisfying the SE axiom and monotonicity? We show, using a variation of the method introduced by Lauwers (2010), that when the domain set (Y) has more than five distinct elements, the existence of any SWO satisfying the SE axiom and monotonicity entails the existence of a non-Ramsey set, a non-constructive object. Thus, if there is no representable social welfare order satisfying the SE axiom and monotonicity, then there is no SWO satisfying the SE axiom and monotonicity which can be constructed.[7]

2 Notation and definitions

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and natural numbers respectively. Let Y, a non-empty subset of $\mathbb{R}$, be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^{\mathbb{N}}$ is the set of all possible utility streams. For all $y, z \in X$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

We consider binary relations on X, denoted by $\succsim$, with symmetric and asymmetric parts denoted by $\sim$ and $\succ$ respectively, defined in the usual way. A *social welfare order* (SWO) is a complete and transitive binary relation. A *social welfare function* (SWF) is a mapping $W : X \to \mathbb{R}$. Given a SWO $\succsim$ on X, we say that $\succsim$ can be *represented* by a real-valued function if there is a mapping $W : X \to \mathbb{R}$ such that for all $x, y \in X$, we have $x \succsim y$ if and only if $W(x) \geq W(y)$.

The SWO that we will be concerned with are required to satisfy an efficiency axiom and an equity axiom. Our efficiency requirement is very weak, and it is called monotonicity. It is difficult to consider a SWO seriously if it violates this axiom.

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7 A similar result holds when the equity concept of finite anonymity is combined with the efficiency concept of weak Pareto. Dubey (2011), who built on and refined results obtained by Lauwers (2010), showed that the existence of a finitely anonymous and weak Paretian ordering (when the cardinality of Y is at least two) implies the existence of a non-Ramsey set. This result can be considered to be one interpretation of the conjecture of Fleurbaey and Michel (2003).

Definition 1 Monotonicity (M): If $x, y \in X$, with $x > y$, then $x \succsim y$.

The equity requirement that we use is called SE; the rationale for it has already been explained in Sect. 1.

Definition 2 Strong Equity (SE): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

It is convenient to define analogous concepts pertaining to a social welfare function $W : X \to \mathbb{R}$.

Definition 3 W-Monotonicity: If $x, y \in X$, with $x > y$, then $W(x) \geq W(y)$.

Definition 4 W-Strong Equity: If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $W(x) > W(y)$.

3 Results

In this section, we present our results on representation, (the possibility result in Proposition 1, the impossibility result in Proposition 2) and construction (Proposition 3) of SWOs satisfying the monotonicity and SE axioms. Together they establish the Theorem stated in the Introduction.

3.1 Representation

First, we construct explicitly a SWF satisfying the SE and monotonicity axioms when the domain set Y consists of a set of five distinct real numbers.

Proposition 1 *Let $Y \equiv \{a, b, c, d, e\}$ be such that $a < b < c < d < e$ and let $X \equiv Y^{\mathbb{N}}$. Given any sequence $x \in X$, define:*

$$N(x) = \{n : n \in \mathbb{N} \text{ and } x_n = a\}, \text{ and } M(x) = \{m : m \in \mathbb{N} \text{ and } x_m = b\}.$$

Let $\alpha(n) = -(1/2^n)$, $\beta(n) = -(1/3^n)$ and $\delta(n) = -\alpha(n)$ for all $n \in \mathbb{N}$. Define the social welfare function $W : X \to \mathbb{R}$ by

$$W(x) = \begin{cases} \sum\limits_{n \in N(x)} \alpha(n) + \sum\limits_{m \in M(x)} \beta(m) & \text{if } N(x) \text{ or } M(x) \text{ is non-empty,} \\ \sum\limits_{n=1}^{\infty} \delta(n)(x_n - a) & \text{otherwise.} \end{cases} \tag{1}$$

Then, $W(x)$ satisfies $W-$SE and $W-$monotonicity.[8]

Proof We first take up $W-$ Strong Equity. Let $x, y \in X$, and let there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. We need to show that $W(x) > W(y)$. There are three cases.

[8] It can be checked that W, defined by (1), also satisfies $W-$ weak Pareto; that is, if $x, y \in X$, and $x \gg y$, then $W(x) > W(y)$.

(a) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that $y_i = a < b = x_i < x_j = c < d = y_j$ or $y_i = a < b = x_i < x_j = c < e = y_j$ or $y_i = a < b = x_i < x_j = d < e = y_j$. Note that $N(x) \cup \{i\} = N(y)$ and $M(x) = M(y) \cup \{i\}$. Then,

$$W(x) = \sum_{n\in N(x)} \alpha(n) + \sum_{n\in M(x)} \beta(n) = \sum_{n\in N(x)} \alpha(n) + \sum_{n\in M(y)} \beta(n) + \beta(i),$$
$$W(y) = \sum_{n\in N(y)} \alpha(n) + \sum_{n\in M(y)} \beta(n) = \sum_{n\in N(x)} \alpha(n) + \alpha(i) + \sum_{n\in M(y)} \beta(n), \text{ and}$$
$$W(x) - W(y) = \beta(i) - \alpha(i) > 0.$$

(b) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that $y_i = a < c = x_i < x_j = d < e = y_j$. Note that $N(x) \cup \{i\} = N(y)$ and $M(x) = M(y)$. So, $W(y) = \sum_{n\in N(y)} \alpha(n) + \sum_{n\in M(y)}\beta(n) = \sum_{n\in N(x)}\alpha(n) + \alpha(i) + \sum_{n\in M(x)}\beta(n) < 0$. For x, the following sub-cases arise:
  (i) either $N(x)$ or $M(x)$ is non-empty. Then, $W(x) = \sum_{n\in N(x)}\alpha(n) + \sum_{n\in M(x)}\beta(n)$, and $W(x) - W(y) = -\alpha(i) > 0$.
  (ii) both $N(x)$ and $M(x)$ are empty. Then, $W(x) = \sum_{n=1}^{\infty}\delta(n)(x_n - a) > 0 > W(y)$.

(c) For $x, y \in X$ let there exist $i, j \in \mathbb{N}$ such that $y_i = b < c = x_i < x_j = d < e = y_j$. Note that $N(x) = N(y)$ and $M(x) \cup \{i\} = M(y)$. So, $W(y) = \sum_{n\in N(y)}\alpha(n) + \sum_{n\in M(y)}\beta(n) = \sum_{n\in N(x)}\alpha(n) + \sum_{n\in M(x)}\beta(n) + \beta(i) < 0$. For x, the following sub-cases arise:
  (i) either $N(x)$ or $M(x)$ is non-empty. Then, $W(x) = \sum_{n\in N(x)}\alpha(n) + \sum_{n\in M(x)}\beta(n)$, and $W(x) - W(y) = -\beta(i) > 0$.
  (ii) both $N(x)$ and $M(x)$ are empty. Then, $W(x) = \sum_{n=1}^{\infty}\delta(n)(x_n - a) > 0 > W(y)$.

Next, we establish $W-$ monotonicity. Let $x, y \in X$ with $x \geq y$. There are again three cases.

(a) Let $y_n > b$ for all $n \in \mathbb{N}$. In this case, $N(x) = M(x) = N(y) = M(y) = \emptyset$. Then, $W(\cdot)$, being sum of discounted one period utilities, clearly satisfies $W-$ monotonicity.

(b) Let $y_n > a$ for all $n \in \mathbb{N}$ and $y_n = b$ for some n. In this case, the sets $N(x) = N(y) = \emptyset$ but $M(y)$ is non-empty. So, $M(x) \subset M(y)$. If $M(x) = \emptyset$, then $W(x) > 0 > W(y)$. If $M(x) \neq \emptyset$, let $N_1 \equiv M(y) \setminus M(x)$. Then, $W(x) - W(y) = -\sum_{n\in N_1}\beta(n) \geq 0$.

(c) Let $y_n = a$ for some $n \in \mathbb{N}$. There are two sub-cases to consider.
  (i) $x_n > b$ for all $n \in \mathbb{N}$. Here, $W(x) > 0 > W(y)$, so $W-$monotonicity is clearly satisfied.
  (ii) $x_n \leq b$ for some $n \in \mathbb{N}$. Here, we define $\bar{N} \equiv \{n \in \mathbb{N} : x_n \neq y_n\}$, $N_2 \equiv \{n \in \bar{N} : y_n = a; x_n = b\}$, $N_3 \equiv \{n \in \bar{N} : y_n = a; x_n > b\}$, and $N_4 \equiv \{n \in \bar{N} : y_n = b\}$. Then, $W(x) - W(y) = \sum_{n\in N_2}(-\alpha(n) + \beta(n)) - \sum_{n\in N_3}\alpha(n) - \sum_{n\in N_4}\beta(n) \geq 0$, establishing $W-$monotonicity.

□

Remark When the domain set Y contains four distinct elements, $Y \equiv \{a, b, c, d\}$ where $a < b < c < d$, the function:

$$W(x) = \begin{cases} \sum_{n \in N(x)} \alpha(n) & \text{if } N(x) \text{ is non-empty,} \\ \sum_{n=1}^{\infty} \delta(n)(x_n - a) & \text{otherwise,} \end{cases} \quad (2)$$

(with notation as in Proposition 1 above) can be shown to satisfy both $W-$ Strong Equity and $W-$monotonicity, by using a proof similar to the proof of Proposition 1.

We can explain informally the nature of the SWF W in (2) above. The key idea in this proof is to separate the "poor" $x_i = a$ from the " non-poor" $x_i \in \{b, c, d\}$, making W entirely insensitive to the utilities of the non-poor generations whenever there is even a single poor generation, and fully sensitive to the presence of each poor generation. It is a standard discounted sum of utilities $(x_i - a)$ when there is no poor generation.

Let us check that W satisfies SE. For $x, y \in X$, let there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$. In this case, we can infer that y_i must be equal to a, and so alternative y has at least one poor generation, namely i. We can also infer that neither generation i nor j is poor in alternative x. Two cases are possible, (i) there is some generation $k \neq i, j$ such that $x_k = a$; and (ii) there is no generation $k \neq i, j$ such that $x_k = a$. In the first case, the first formula in (2) applies to both x and y, and $W(x) > W(y)$ because y has one more poor generation than x. It is easy to check the second case.

Next we check that W satisfies monotonicity. Take $x, y \in X$, such that $x \geq y$. There are three possibilities, (i) both x and y have a poor generation; (ii) neither x nor y has a poor generation; and (iii) y has a poor generation but x does not. In the first case, the first formula in (2) applies to both x and y, and so $W(x) \geq W(y)$. The other two cases can be verified similarly.

Proposition 2 *If Y contains more than five distinct elements, then there does not exist a representable social welfare order on $X \equiv Y^{\mathbb{N}}$, satisfying the Strong Equity axiom and monotonicity.*

Proof Suppose, on the contrary, there is $Y = \{a, b, c, d, e, f\}$, where $a < b < c < d < e < f$, and $\succsim$ is a representable social welfare order on $X = Y^{\mathbb{N}}$ satisfying the Strong Equity axiom and monotonicity. Let $W : X \to \mathbb{R}$ be a function which represents $\succsim$ on X.

Let $I \equiv (0, 1)$ and $\{r_1, r_2, \ldots\}$ be a given enumeration of the rational numbers in I. For each real number $p \in I$, define $N(p) = \{n : n \in \mathbb{N}; n > 2 : r_n \in (0, p)\}$ and $M(p) = \mathbb{N} \setminus \{N(p) \cup \{1, 2\}\}$. Define following pair of sequences $x(p) \in X$ and $y(p) \in X$ as:

$$x_n(p) = \begin{cases} f & \text{if } n = 1, \\ c & \text{if } n = 2, \\ b & \text{if } n \in N(p), \\ a & \text{otherwise,} \end{cases} \qquad y_n(p) = \begin{cases} e & \text{if } n = 1, \\ d & \text{if } n = 2, \\ b & \text{if } n \in N(p), \\ a & \text{otherwise,} \end{cases} \quad (3)$$

Note that $x_2(p) = c < d = y_2(p) < y_1(p) = e < f = x_1(p)$ and $x_n(p) = y_n(p)$ for all $n > 2$. Hence by SE, $y(p) \succ x(p)$, and $W(y(p)) > W(x(p))$. Now let $q \in (p, 1)$. Observe that $N(p) \subset N(q)$ and $M(q) \subset M(p)$. There are infinitely many elements in $N(q) \cap M(p)$. Let $j(p, q) \equiv \min\{N(q) \cap M(p)\}$ for which $y_{j(p,q)}(p) = a < b = x_{j(p,q)}(q)$ holds. Then $y_{j(p,q)}(p) = a < b = x_{j(p,q)}(q) < c = x_2(q) < d = y_2(p)$; and $x_n(q) \geq y_n(p)$ for all other $n \in \mathbb{N}$. This implies $x(q) \succ y(p)$ by SE and monotonicity and so $W(x(q)) > W(y(p))$. This leads to a contradiction, by using the arguments in (Basu and Mitra 2003, Theorem 1). □

3.2 Construction

Let T be an infinite subset of $\mathbb{N}$. We denote by $\Omega(T)$ the collection of all infinite subsets of T, and we denote $\Omega(\mathbb{N})$ by Ω. Thus, for any infinite subset T of $\mathbb{N}$, we have $T \subset \mathbb{N}$, and $T \in \Omega$.

Definition 5 Ramsey Collection of Sets: A collection of sets $\Gamma \subset \Omega$ is called Ramsey if there exists $T \in \Omega$ such that either $\Omega(T) \subset \Gamma$ or $\Omega(T) \subset \Omega \setminus \Gamma$.

If a collection of sets $\Gamma \subset \Omega$ does not have the property stated in Definition 5 (so that for every $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and its complement $\Omega \setminus \Gamma$) then the collection of sets Γ is called non-Ramsey.[9]

We will say that a collection of sets $\Gamma \subset \Omega$ can be *constructed* if it can be obtained in *every* model of set theory, satisfying the Zermelo-Frankel axioms. If a collection of sets $\Gamma \subset \Omega$ cannot be constructed, we call it a *non-constructive* object.

In a seminal paper, Solovay (1970) obtained a model of set theory, satisfying the Zermelo-Frankel axioms, in which the Axiom of Choice is false and every set of real numbers is Lebesgue measurable.[10] Subsequently, Mathias (1977) showed that in Solovay's model every collection of sets $\Gamma \subset \Omega$ is Ramsey. This implies that any collection of sets $\Gamma \subset \Omega$ which is non-Ramsey cannot be obtained in Solovay's model and so, according to our definition, cannot be constructed.[11] Thus, a non-Ramsey collection of sets $\Gamma \subset \Omega$ is a non-constructive object.

Our principal result is that when the domain set (Y) has more than five distinct elements, the existence of any SWO satisfying the SE axiom and monotonicity implies the existence of a non-Ramsey set, a non-constructive object.[12]

Proposition 3 *Let Y consists of more than five distinct elements, and let $X \equiv Y^{\mathbb{N}}$. The existence of a social welfare order on X, which satisfies the Strong Equity axiom and monotonicity, entails the existence of a non-Ramsey set.*

[9] If one considers Ω to be a topological space, endowed with the standard product topology, then any Borel subset of Ω is Ramsey (Galvin and Prikry (1973)). On the other hand, (Erdós and Rado 1952, Example 1, p. 434) have shown, using Zermelo's well-ordering principle (which is known to be equivalent to the Axiom of Choice), that there is a collection of sets $\Gamma \subset \Omega$, which is non-Ramsey.

[10] Solovay's model also satisfies the Axiom of Dependent Choice, which in turn implies the Axiom of Countable Choice.

[11] In particular, the non-Ramsey collection of sets obtained by Erdós and Rado (1952) cannot be constructed.

[12] For an informal presentation of the principal idea of the proof of this result, the reader can consult (Dubey and Mitra 2013a, p. 6).

Proof Define $Y \equiv \{a, b, c, d, e, f\}$, with $a < b < c < d < e < f$. Let $N \equiv \{n_1, n_2, n_3, n_4, \ldots\}$ be an infinite subset of $\mathbb{N}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Let $\overline{N} = \{1, 2, \ldots, 2(n_4 - 1)\}$. For any $T \in \Omega(N)$, $T \equiv \{t_1, t_2, t_3, t_4, \ldots\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}$, we partition the set of natural numbers $\mathbb{N}$ in $U = \{2t_1 - 1, 2t_1, \ldots, 2(t_2 - 1), 2t_3 - 1, \ldots, 2(t_4 - 1), \ldots\}$ and $L = \mathbb{N} \setminus U = \{1, 2, \ldots, 2(t_1 - 1), 2t_2 - 1, 2t_2, \ldots, 2(t_3 - 1), \ldots\}$. Let $\overline{LTE} = \{t \in L \cap \overline{N} : t \text{ is even}\}$ and $\overline{LTO} = L \cap \overline{N} \setminus \overline{LTE}$. Also, $\overline{UTE} = \{t \in U \cap \overline{N} : t \text{ is even}\}$, $\overline{UTO} = U \cap \overline{N} \setminus \overline{UTE}$, $\overline{LCN} = L \setminus \overline{N}$, and $\overline{UCN} = U \setminus \overline{N}$. We define the utility stream $x(T, \overline{N})$ whose components are,

$$x_t = \begin{cases} c \text{ if } t \in \overline{LTO}, & f \text{ if } t \in \overline{LTE}, \\ d \text{ if } t \in \overline{UTO}, & e \text{ if } t \in \overline{UTE}, \\ a \text{ if } t \in \overline{LCN}, & b \text{ if } t \in \overline{UCN}. \end{cases} \tag{4}$$

We also define the sequence $y(T, \overline{N})$ using the subset $T \setminus \{t_1\}$ in place of subset T, in the following fashion. The two partitions of the set of natural numbers $\mathbb{N}$ are $\widehat{U} = \{2t_2 - 1, 2t_2, \ldots, 2(t_3 - 1), 2t_4 - 1, \ldots, 2(t_5 - 1), \ldots\}$ and $\widehat{L} = \mathbb{N} \setminus U$. Let $\widehat{LTE} = \{t \in \widehat{L} \cap \overline{N} : t \text{ is even}\}$ and $\widehat{LTO} = \widehat{L} \cap \overline{N} \setminus \widehat{LTE}$. Also, $\widehat{UTE} = \{t \in \widehat{U} \cap \overline{N} : t \text{ is even}\}$, $\widehat{UTO} = \widehat{U} \cap \overline{N} \setminus \widehat{UTE}$, $\widehat{LCN} = \widehat{L} \setminus \overline{N}$, and $\widehat{UCN} = \widehat{U} \setminus \overline{N}$. We define the utility stream $y(T, \overline{N})$ whose components are,[13]

$$y_t = \begin{cases} c \text{ if } t \in \widehat{LTO}, & f \text{ if } t \in \widehat{LTE}, \\ d \text{ if } t \in \widehat{UTO}, & e \text{ if } t \in \widehat{UTE}, \\ a \text{ if } t \in \widehat{LCN}, & b \text{ if } t \in \widehat{UCN}. \end{cases} \tag{5}$$

As $\overline{N}$ is unique for any N, $x(S, \overline{N})$ and $y(S, \overline{N})$ are well-defined for any $S \in \Omega(N)$.

Let $\succsim$ be a SWO satisfying monotonicity and SE. We claim that the collection of sets $\Gamma \equiv \{N \in \Omega : y(N) \succ x(N)\}$ is non-Ramsey. We need to show that for each $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and $\Omega \diagdown \Gamma$. For this, it is sufficient to show that for each $T \in \Omega$, there exists $S \in \Omega(T)$ such that either $T \in \Gamma$ or $S \in \Gamma$, with the either/or being exclusive. Let $T \equiv \{t_1, t_2, \ldots\}$. In the remaining proof we are concerned with infinite utility sequences $x(T, \overline{T})$, $y(T, \overline{T})$ and $x(S, \overline{T})$, $y(S, \overline{T})$ where $S \in \Omega(T)$. For ease of notation, we omit reference to $\overline{T}$. As the binary relation is complete, one of the following cases must arise: (a) $y(T) \succ x(T)$; (b) $x(T) \succ y(T)$; (c) $x(T) \sim y(T)$. Accordingly, we now separate our analysis into three cases.

[13] If $n_1 = 1$, then $\{1, \ldots, 2(n_1 - 1)\} = \emptyset$. For illustration, for $N = \{1, 2, 3, 4, \ldots\}$, $\overline{N} = \{1, 2, \ldots, 6\}$ and the two utility streams are $x(N, \overline{N}) = \{d, e, c, f, d, e, a, a, b, b, \ldots\}$ and $y(N, \overline{N}) = \{c, f, d, e, c, f, b, b, a, a, \ldots\}$.

(a) Let $y(T) \succ x(T)$, i.e., $T \in \Gamma$. We drop t_1 from T to obtain $S = \{t_2, t_3, t_4, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\}$ and $T_2 \equiv \{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\}$. Observe that
  (A) for all $t \in \mathbb{N}$, $x_t(S) = y_t(T)$;
  (B) for all $t \in T_1$, $x_t(T) = d > c = y_t(S)$; for all $t \in T_2$, $x_t(T) = e < f = y_t(S)$; and
  (C) for all the remaining $t \in \mathbb{N}$, $x_t(T) = y_t(S)$.

  Then for the generations $2t_1 - 1$ and $2t_1$, $y_{2t_1-1}(S) = c < d = x_{2t_1-1}(T) < x_{2t_1}(T) = e < f = y_{2t_1}(S)$. Similar inequalities hold for the pair of generations $\{2t_1 + 1, 2t_1 + 2\}, \ldots, \{2t_2 - 3, 2t_2 - 2\}$. Each of these pairs leads to SE improvements in $x(T)$ compared to $y(S)$. Since these are finitely many SE improvements, $x(T) \succ y(S)$ by SE. Also, $x(S) \sim y(T)$. Since $y(T) \succ x(T)$, we get, $x(S) \sim y(T) \succ x(T) \succ y(S)$. Thus, $x(S) \succ y(S)$ by transitivity of $\succsim$, and so $S \notin \Gamma$.

(b) Let $x(T) \succ y(T)$, i.e., $T \notin \Gamma$. We drop t_1 and t_{4n}, t_{4n+1} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_6, t_7, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\}$, $T_2 \equiv \{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\}$, and $\widehat{T} \equiv \{2t_{4n} - 1, 2t_{4n}, \ldots, 2t_{4n+1} - 2 : n \in \mathbb{N}\}$. Observe that,
  (A) for all $t \in T_1$, $x_t(T) = d > c = y_t(S)$; for all $t \in T_2$, $x_t(T) = e < f = y_t(S)$;
  (B) for all $t \in \widehat{T}$, $x_t(T) = x_t(S) = a < b = y_t(T) = y_t(S)$; and
  (C) for all the remaining coordinates, $x_t(T) = y_t(S)$ and $x_t(S) = y_t(T)$.

  Then for the generations $2t_1 - 1$ and $2t_4 - 1$, $x_{2t_4-1}(T) = a < b = y_{2t_4-1}(S) < y_{2t_1-1}(S) = c < d = x_{2t_1-1}(T)$. There are finitely many generations in T_1 and infinitely many generations in $\widehat{T}$. Let the cardinality of set T_1 be K. Thus it is possible to choose generations $l_1 = 2t_4 - 1, l_2, \ldots, l_K$ from $\widehat{T}$ such that similar inequalities hold for the pair of generations $\{2t_1 + 1, l_2\}, \ldots, \{2t_2 - 3, l_K\}$. Each of these pairs leads to SE improvements in $y(S)$ compared to $x(T)$. Since these are finitely many SE improvements, and also by comparing remaining generations $t \in T_2 \cup \hat{T} \setminus \{l_1, \ldots, l_K\}$, $y(S) \succ x(T)$ by SE and M. Also, $y(T) \succsim x(S)$ by M. Since $x(T) \succ y(T)$, we get $y(S) \succ x(T) \succ y(T) \succsim x(S)$. Thus, $y(S) \succ x(S)$ by transitivity of $\succsim$, and so $S \in \Gamma$.

(c) Let $x(T) \sim y(T)$, i.e., $T \notin \Gamma$. We drop t_1, t_2, t_3 and t_{4n+2}, t_{4n+3} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_4, t_5, t_8, t_9, \ldots\}$. Hence $S \in \Omega(T)$. Denote the set of coordinates $T_1 \equiv \{2t_2 - 1, 2t_2 + 1, \ldots, 2t_3 - 3\}$, $T_2 \equiv \{2t_2, 2t_2 + 2, \ldots, 2t_3 - 2\}$, $T_3 \equiv \{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\} \cup \{2t_3 - 1, 2t_3 + 1, \ldots, 2t_4 - 3\}$, $T_4 \equiv \{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\} \cup \{2t_3, 2t_3 + 2, \ldots, 2t_4 - 2\}$, and $\widehat{T} \equiv \{2t_{4n+2} - 1, \ldots, 2t_{4n+3} - 2 : n \in \mathbb{N}\}$.
  (i) For $x(S)$ and $y(T)$,
    (A) for all $t \in T_1$, $y_t(T) = d > c = x_t(S)$; for all $t \in T_2$, $y_t(T) = e < f = x_t(S)$;
    (B) for all $t \in \widehat{T}$, $x_t(S) = a < b = y_t(T)$; and
    (C) for all the remaining coordinates, $y_t(T) = x_t(S)$.

Then for the generations $2t_2 - 1$ and $2t_2$, $x_{2t_2-1}(S) = c < d = y_{2t_2-1}(T) < y_{2t_2}(T) = e < f = x_{2t_2}(S)$. Similar inequalities hold for the pair of generations $\{2t_2 + 1, 2t_2 + 2\}, \ldots, \{2t_3 - 3, 2t_3 - 2\}$. Each of these pairs leads to SE improvements in $y(T)$ compared to $x(S)$. Since these are finitely many pairs of SE improvements, and also by comparing generations $t \in \widehat{T}$, $x(S) \prec y(T)$ by SE and M.

(ii) For $x(T)$ and $y(S)$,

(A) for all $t \in T_3$, $x_t(T) = d > c = y_t(S)$; for all $t \in T_4$, $x_t(T) = e < f = y_t(S)$;

(B) for all $t \in \widehat{T}$, $x_t(T) = a < b = y_t(S)$; and

(C) for all the remaining coordinates, $x_t(T) = y_t(S)$.

Then for the generations $2t_1 - 1$ and $2t_6 - 1$, $x_{2t_6-1}(T) = a < b = y_{2t_6-1}(S) < y_{2t_1-1}(S) = c < d = x_{2t_1-1}(T)$. There are finitely many generations in T_3 and infinitely many generations in $\widehat{T}$. Let the cardinality of set T_3 be K. Thus it is possible to choose generations $l_1 = 2t_6 - 1, l_2, \ldots, l_K$ from $\widehat{T}$ such that similar inequalities hold for the pair of generations $\{2t_1 + 1, l_2\}, \ldots, \{2t_4 - 3, l_K\}$. Each of these pairs leads to SE improvements in $y(S)$ compared to $x(T)$. Since these are finitely many pairs of SE improvements, and also by comparing remaining generations $t \in T_4 \cup \widehat{T} \setminus \{l_1, \ldots, l_K\}$, $y(S) \succ x(T)$ by SE and M.

Since $x(T) \sim y(T)$, we get $y(S) \succ x(T) \sim y(T) \succ x(S)$. Thus, $y(S) \succ x(S)$ by transitivity of $\succsim$, and so $S \in \Gamma$. □

4 Conclusions

In this paper we have characterized the domain restrictions for representable monotone SWO satisfying SE axiom. Together with results available in the literature, this enables us to provide the following useful summary of the results on representable SWOs, satisfying Hammond Equity or SE, when these distributive equity principles are combined with some standard efficiency principles (Table 1).

Table 1 Representation, efficiency, Hammond Equity and Strong Equity

	Strong Pareto	Monotonicity
Hammond Equity	Trivial possibility (≤ 3 elements) Impossibility (> 3 elements)	Trivial possibility
Strong Equity	Trivial possibility (≤ 3 elements) Impossibility (> 3 elements)	Possibility (≤ 5 elements) Impossibility (> 5 elements)

In addition to the results noted in the table above, we note that Alcantud and Garcia-Sanz (2013) have established a possibility result on the representation of SWO, satisfying Hammond Equity and Weak Pareto. Dubey and Mitra (2013c) have recently completely characterized the domains Y for which such a representation result is possible.

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